EXACTNESS IN THE PATH INTEGRAL OF THE COULOMB POTENTIAL IN ONE SPACE DIMENSION

SEJII SAKODA
Department of Applied Physics, National Defense Academy
Hashirimizu, Yokosuka city, Kanagawa 239-8686, Japan
sakoda@nda.ac.jp

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Abstract

We solve time-sliced path integrals of one-dimensional Coulomb system in an exact manner. In formulating path integrals, we make use of the Duru-Kleinert transformation with Fujikawa's gauge theoretical technique. Feynman kernels in the momentum representation both for bound states and scattering states will be obtained with clear pole structure that explains the exactness of the path integral. The path integrals presented here can be, therefore, evaluated exactly by making use of Cauchy's integral theorem.

1 Introduction

One-dimensional system with the Coulomb potential, or the hydrogen atom in one space dimension, may find its application not so much in practice.[1] Nevertheless, after a work by Loudon[1], a number of publications on this apparently simple system have appeared[2]–[6] to discuss the peculiarity of the singular potential. The strange result obtained by Loudon was the twofold degeneracy in the energy eigenvalues despite the one-dimensional nature of the system. Another interesting but quite controversial issue was the (non-)existence[2, 3, 4] of the ground state of even parity with infinite binding energy. Exclusion of the even states with finite energy[5] and relativistic effect for the removal of such unphysical states[6] were also argued. Among these works, it seems that only a few attempts on analyses from the path integral formalism have so far been made. To our knowledge, only in Ref.[7] descriptions of functional-integral treatment, in addition to the functional-analytic approach, of the one-dimensional hydrogen atom can be found. The technique, based on the probabilistic measure, of the functional-integral employed in Ref.[7] is rather mathematical than that we wish to establish in this article. Furthermore the actual calculation of the functional-integral is worked out by making use of some identity that connects the Green function of the one-dimensional hydrogen atom to that of a four-dimensional harmonic oscillator. In other words, the path integral of the one-dimensional hydrogen atom is not still solved explicitly. It cannot be, therefore, sufficient for our purpose to understand the exactness of the path integral for one-dimensional hydrogen atom by such calculation. As stated in Ref.[7], there may exist some natural prescription for the boundary condition of wave functions at the origin required from the consistency of the path integral. We should be able to find such structure by constructing path integrals in the time-sliced form keeping connection with the operator formalism. Therefore, from theoretical as well as pedagogical points of view, it will be useful to develop another path integral technique for solving such a singular system.

Path integral of the hydrogen atom was first successfully formulated by Duru and Kleinert[8, 9] by making use of the so called Duru-Kleinert(DK) transformation. The DK transformation consists of two fundamental ingredients: reparametrization of the time in the path integral and making use of the "square root of the coordinates" by means of a Kustaanheimo-Stiefel(KS) transformation[10]. Many attempts, mainly on the use of KS transformation, by which we can

[1] After the completion of this work, the author has been made aware of experimental studies on the exciton spectra[24] under the high magnetic field. It is shown in Ref.[25] that an exciton under extremely high magnetic field can be well approximated by the one-dimensional Coulomb system.

write the Coulomb path integral in terms of oscillator coordinates, with or without time-slicing, have been made to clarify the approach by DK transformation[11]–[22]. It was Fujikawa[23] who first observed that the essence of the DK transformation is the special choice of the gauge fixing condition for a system with invariance under reparametrization of the time. He proposed a new approach to formulate a Coulomb path integral in an exactly solvable way from the view point of the Jacobi's principle of least action. He also noticed that the use of KS transformation in the Coulomb path integral is rather technical and demonstrated the exact solubility of the Coulomb path integral in three dimension in terms of the parabolic coordinates.

The hydrogen atom in one space dimension seems to be rather exceptional in the path integral formalism. Although many examples with detailed proofs are found in the textbook[22] by Kleinert, the space dimension $D_{\mathcal{C}}$ for the path integral of Coulomb potential in chapter 14 of the Ref.[22] is assumed to be $D_{\mathcal{C}} \geq 2$. In particular, if we set $D_{\mathcal{C}} = 1$ in Eq.(14.93) of Ref.[22], we obtain $D_{\mathcal{O}} = 0$ as the space dimension of the corresponding, or DK-equivalent, oscillator coordinates. In fact, there does not seem to be known any KS-type transformation for one-dimensional Coulomb system excepting the one utilized in Ref.[7] as mentioned above. Motivated by these observations, we adopt in this article Fujikawa's formulation to find path integral descriptions for one-dimensional Coulomb potential $V(x) = -\alpha/r$, where $r = \sqrt{x^2}$. Use of the KS transformation will be thus avoided in our formulation. It was the problem of operator ordering, arising from nonlinear canonical transformations such as KS transformation, in path integrals that forced some authors to criticize the DK method with KS transformation(see Ref.[20], [22] and references therein). Our formulation of time-sliced path integrals in this article will be built basing on a symmetric Hamiltonian operator that has definite rule of operator ordering, thereby being free from the ordering problem. The effect of time-slicing from the formal continuum path integral will also be problematic if we rely on such formal derivation of path integrals. In our formalism, the basic ingredient is the short-time kernel that describes an infinitesimal time evolution. Feynman kernels of finite time evolution will be constructed by multiplication of such infinitesimal kernels. Without this process it will be impossible to understand the mechanism of exactness in path integrals considered here.

We organize this paper as follows: in section 2 we will briefly explain Fujikawa's method then formulate immediately a path integral with a negative fixed energy in the Euclidean formalism. For the case of positive fixed energy, the Euclidean technique does not work well. Hence it will be treated separately for the usual Feynman kernel by setting the time parameter to be real. For both of these path integrals we shall develop a novel procedure in multiplication of infinitesimal kernels in the momentum representation. The use of the momentum space wave functions is due to the form of the symmetric Hamiltonian we set up. This will become clear soon in the next section. Section 3 will be devoted to the construction of the physical states from the solution of path integrals in section 2. In Fujikawa's procedure this is the important step for solving the problem in terms of path integral. This will be achieved by analyses of the resolvent operator. Conclusions and remarks will be found in the final section.

2 Path Integrals of the Free Particle with a Fixed Energy

For a system, whose Lagrangian being given by $L_{\text{orig}} = \dot{x}^2/2 - V(x)$, we consider another Lagrangian with reparametrization invariance, given by

$$L = \sqrt{2(E - V(x))\left(\frac{dx}{dt}\right)^2} \tag{1}$$

for a fixed energy E, according to the procedure introduced by Fujikawa in Ref. [23]. The invariance of this new system under a time reparametrization requires a gauge fixing upon quantization. As was shown by Fujikawa, by choosing an appropriate gauge condition, we can formulate a path integral that is equivalent to the DK transform of the path integral for the original system.

The Lagrangian (1) yields the canonical momentum, given by

$$p = \sqrt{2(E - V(x))}\,\frac{dx}{dt}\Bigg/\sqrt{\left(\frac{dx}{dt}\right)^2}\,, \tag{2}$$

that results in a constraint

$$\phi_0 := \frac{1}{2}p^2 + V(x) - E = 0. \tag{3}$$

In addition, due to the fact $H = p\dot{x} - L = 0$, we need to test Dirac's total Hamiltonian, given by

$$H_{\rm T} = u(x,p)\phi_0 \approx 0, \tag{4}$$

in which the Lagrange multiplier $u(x,p)$ cannot be determined from the equation of motion at all. Choosing this rather arbitrary degree by hand exactly corresponds to the choice of gauge fixing condition. The convenient way for us is setting $u(x,p) = r$ to get

$$H_{\rm T} = \frac{1}{2}rp^2 - \alpha - Er, \tag{5}$$

for the one-dimensional Coulomb potential. However, there is no restriction in application of this procedure and gauge fixing to systems other than the Coulomb system. We may, for example, enjoy this method even for the free particle by setting $\alpha = 0$ in the above. Furthermore, if we solve path integrals of the free particle in this formalism, the solution of corresponding ones for the Coulomb system is immediate, since the difference in the Hamiltonian $H_{\rm T}$ of $\alpha \neq 0$ from that of $\alpha = 0$ is merely a constant. We therefore first examine the quantization and its path integral of the total Hamiltonian (5) for the case of $\alpha = 0$ in the following.

2.1 Path integral for the negative energy

Quantization of the system can be achieved by adopting a unitary representation of the canonical commutation relation(CCR)

$$[x,p] = i \tag{6}$$

with the constraint (4) as a physical state condition. Since the Hamiltonian $H_{\rm T}$ involves a term that consists of r multiplied by p^2, we need to specify an operator ordering upon quantization. For brevity we choose a symmetric product for this term to consider the operator Hamiltonian

$$\hat{H}_{\rm S} = \frac{1}{2}\hat{p}\hat{r}\hat{p} - E\hat{r}. \tag{7}$$

To elucidate the property of this Hamiltonian, we here put $E = -\omega^2/2$ and evaluate a Euclidean kernel(a Feynman kernel with imaginary time) in the momentum representation:

$$\tilde{K}(p_b,\tau_b|p_a,\tau_a) = \langle p_b|e^{-\hat{H}_{\rm S}(\tau_b-\tau_a)}|p_a\rangle, \tag{8}$$

where $|p\rangle$ designates the momentum eigenvector. Since the Hamiltonian (7) is linear in $\hat{r}$, the integration over x in a path integral can be easily performed. This is the reason why we take the momentum(p) representation instead of the coordinate(x) representation.

The first step for the evaluation of the kernel is the calculation of an infinitesimal kernel

$$\langle p|\left(1-\epsilon\hat{H}_{\rm S}\right)|p'\rangle = \int_{-\infty}^{\infty}\frac{dx}{2\pi}\,\exp\left[-i(p-p')x - \frac{r\epsilon}{2}\left(pp'+\omega^2\right)\right], \tag{9}$$

where $\epsilon = (\tau_b - \tau_a)/N > 0$ is taken to be $\omega\epsilon \ll 1$. We split the integration domain at $x = 0$ and put $x \to -x$ for negative x to find

$$\langle p|\left(1-\epsilon\hat{H}_{\rm S}\right)|p'\rangle = \sum_{\sigma=\pm1}\int_0^{\infty}\frac{dx}{2\pi}\,\exp\left[-\left\{i(p-p')\sigma + \frac{\epsilon}{2}\left(pp'+\omega^2\right)\right\}x\right]. \tag{10}$$

To ensure the convergence of this integration, a condition, $pp' + \omega^2 > 0$, is necessary. Although it is not explicit, this condition is equivalent to $p^2 + \omega^2 > 0 (p'^2 + \omega^2 > 0)$ that holds always for real $p(p')$. To check this we rewrite pp' as $p^2 - p(p-p')$ in the above to make a change of variable from x to $(1 + i\epsilon p\sigma/2)x$. By taking the Jacobian into account, we find that the right hand side of Eq.(10) is equivalent to

$$\sum_{\sigma=\pm1}\int_0^{\infty}\frac{dx}{2\pi}\,\exp\left[\frac{i\epsilon}{2}p\sigma - \left\{i(p-p')\sigma + \frac{\epsilon}{2}\left(p^2+\omega^2\right)\right\}x\right], \tag{11}$$

up to $O(\epsilon)$ in the exponent. Note that the additional term from the Jacobian precisely matches to the one we need when rewriting the symmetric product $\hat{p}\hat{r}\hat{p}$ as $\hat{p}^2\hat{r} + [\hat{r}, \hat{p}]$. The condition for convergence of the integration now reads $p^2 + \omega^2 > 0$ for this new expression. This proves the above statement. We can therefore proceed to carry out the integration with respect to x to find

$$\langle p| \left(1 - \epsilon \hat{H}_{\rm S}\right) |p'\rangle = \frac{1}{2\pi i} \left[\frac{1}{p - p' - i\epsilon \left(pp' + \omega^2\right)/2} - \frac{1}{p - p' + i\epsilon \left(pp' + \omega^2\right)/2} \right]. \tag{12}$$

Once we obtain the expression (12), we can regard it as a function of two complex variables p and p'. As a function of p, each term in the right hand side has a simple pole at

$$p = \frac{p' + i\epsilon\omega^2/2}{1 - i\epsilon p'/2}, \ \frac{p' - i\epsilon\omega^2/2}{1 + i\epsilon p'/2}, \tag{13}$$

respectively. As a confirmation, we observe that the real part of $pp' + \omega^2$ is positive for both of these poles if we set p' to be real in this evaluation. It will be interesting to see that the situation is quite similar to the one for an expression of a delta function

$$\delta(p - p^{'}) = \frac{1}{2\pi i} \left(\frac{1}{p - p' - i0_+} - \frac{1}{p - p' + i0_+} \right), \tag{14}$$

where 0_+ designates a positive infinitesimal. We may often utilize this notation hereafter.

The second step for us to proceed is the multiplication of the infinitesimal kernels. But before doing this we observe

$$p - p' \mp i\frac{\epsilon}{2} \left(pp' + \omega^2\right) = (p - p') \cosh(\omega\epsilon/2) \mp \frac{i}{\omega} \left(pp' + \omega^2\right) \sinh(\omega\epsilon/2) + O(\epsilon^2) \tag{15}$$

so that we can imagine the kernel for a finite imaginary time $\tau (\tau = \tau_b - \tau_a)$ to be

$$\tilde{K}(p_b, \tau_b | p_a, \tau_a) = \sum_{\sigma = \pm 1} \frac{1}{2\pi i} \frac{\sigma}{(p_b - p_a) \cosh(\omega\tau/2) - i\frac{\sigma}{\omega} \left(p_b p_a + \omega^2\right) \sinh(\omega\tau/2)}. \tag{16}$$

Let us prove this by explicitly evaluating a product of infinitesimal kernels. To this aim we recall the analogy to the delta function above again. The first term of the infinitesimal kernel (12) possesses a simple pole of positive(negative) imaginary part as a function of $p(p')$ while the situation switches to the opposite in the second term. We can therefore make use of the Cauchy's integral formula, just like the case we will do in the multiplication of two delta functions, in carrying out the integration

$$\langle p_2| \left(1 - \epsilon \hat{H}_{\rm S}\right)^2 |p_0\rangle = \int_{-\infty}^{\infty} dp_1 \, \langle p_2| \left(1 - \epsilon \hat{H}_{\rm S}\right) |p_1\rangle \langle p_1| \left(1 - \epsilon \hat{H}_{\rm S}\right) |p_0\rangle, \tag{17}$$

to obtain

$$\langle p_2| \left(1 - \epsilon \hat{H}_{\rm S}\right)^2 |p_0\rangle = \frac{1}{2\pi i} \times \left[\frac{1}{(p_2 - p_0)\{1 + (\omega\epsilon/2)^2\} - i\omega\epsilon \frac{1}{\omega} \left(p_2 p_0 + \omega^2\right)} \right.$$
$$\left. - \frac{1}{(p_2 - p_0)\{1 + (\omega\epsilon/2)^2\} + i\omega\epsilon \frac{1}{\omega} \left(p_2 p_0 + \omega^2\right)} \right]. \tag{18}$$

This suggests us to suppose

$$\langle p_j| \left(1 - \epsilon \hat{H}_{\rm S}\right)^j |p_0\rangle = \frac{1}{2\pi i}$$
$$\times \left[\frac{1}{(p_j - p_0) C_j - \frac{i}{\omega} S_j \left(p_j p_0 + \omega^2\right)} - \frac{1}{(p_j - p_0) C_j + \frac{i}{\omega} S_j \left(p_j p_0 + \omega^2\right)} \right], \tag{19}$$

with initial conditions, $C_0 = 1$ and $S_0 = 0_+$. Then, by repeating the same procedure above, we find that one more multiplication by a short-time kernel $\langle p_{j+1}| \left(1 - \epsilon \hat{H}_{\rm S}\right) |p_j\rangle$ to this expression results in

$$\begin{aligned}\langle p_{j+1}| \left(1 - \epsilon \hat{H}_{\rm S}\right)^{j+1} |p_0\rangle &= \frac{1}{2\pi i} \\ \times \sum_{\sigma=\pm 1} &\frac{\sigma}{(p_{j+1} - p_0)(C_j + \omega\epsilon S_j/2) - i\frac{\sigma}{\omega}(\omega\epsilon C_j/2 + S_j)\,(p_{j+1}p_0 + \omega^2)}.\end{aligned} \tag{20}$$

By comparing Eq.(20) with Eq.(19), we obtain a set of recurrence formulas

$$C_{j+1} = C_j + \frac{\omega\epsilon}{2} S_j, \quad S_{j+1} = \frac{\omega\epsilon}{2} C_j + S_j. \tag{21}$$

The solution of the above can be found immediately to be

$$C_j = \cosh(j\omega\epsilon/2), \quad S_j = \sinh(j\omega\epsilon/2) \tag{22}$$

to convince us that our conjecture in Eq.(16) holds.

Eigenvalues and eigenfunctions of the Hamiltonian $\hat{H}_{\rm S}$ can be extracted from the kernel in Eq.(16). To see this we rewrite the first term of $\tilde{K}(p_b, \tau_b | p_a, \tau_a)$ as follows

$$\begin{aligned}&\frac{1}{2\pi i} \frac{1}{(p_b - p_a)\cosh(\omega\tau/2) - \dfrac{i}{\omega}\,(p_b p_a + \omega^2)\sinh(\omega\tau/2)} \\ =&\frac{\omega}{\pi} \frac{1}{e^{\omega\tau/2}(\omega + ip_b)(\omega - ip_a) - e^{-\omega\tau/2}(\omega - ip_b)(\omega + ip_a)} \\ =&\frac{\omega}{\pi} \frac{1}{(\omega + ip_b)(\omega - ip_a)} \sum_{n=0}^{\infty} e^{-(n+1/2)\omega\tau} \left(\frac{\omega - ip_b}{\omega + ip_b}\right)^n \left(\frac{\omega + ip_a}{\omega - ip_a}\right)^n.\end{aligned} \tag{23}$$

In the same way, the second term is also expanded in a series as

$$\begin{aligned}&-\frac{1}{2\pi i} \frac{1}{(p_b - p_a)\cosh(\omega\tau/2) + \dfrac{i}{\omega}\,(p_b p_a + \omega^2)\sinh(\omega\tau/2)} \\ =&\frac{\omega}{\pi} \frac{1}{(\omega - ip_b)(\omega + ip_a)} \sum_{n=0}^{\infty} e^{-(n+1/2)\omega\tau} \left(\frac{\omega + ip_b}{\omega - ip_b}\right)^n \left(\frac{\omega - ip_a}{\omega + ip_a}\right)^n.\end{aligned} \tag{24}$$

We thus obtain another expression of the kernel in the series of eigenfunctions, given by

$$\tilde{K}(p_b, \tau_b | p_a, \tau_a) = \sum_{n=0}^{\infty} e^{-(n+1/2)\omega\tau} \left\{ \tilde{\psi}_n^{(+)}(p_b) \tilde{\psi}_n^{(+)}(p_a)^* + \tilde{\psi}_n^{(-)}(p_b) \tilde{\psi}_n^{(-)}(p_a)^* \right\}, \tag{25}$$

where $\tilde{\psi}_n^{(\pm)}(p)$ for $n = 0, 1, 2, \ldots$, defined as

$$\tilde{\psi}_n^{(\pm)}(p) = \sqrt{\frac{\omega}{\pi}} \frac{1}{\omega \pm ip} \left(\frac{\omega \mp ip}{\omega \pm ip}\right)^n, \tag{26}$$

are eigenfunctions belonging to a common eigenvalue $\lambda_n^{(\rm S)}(E) = (n + 1/2)\omega$ in the momentum representation. From these manipulation we observe the twofold degeneracy in all eigenvalues of the Hamiltonian. Eigenfunctions in the position representation can be obtained from $\tilde{\psi}_n^{(\pm)}(p)$ as

$$\psi_n^{(\pm)}(x) = \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi}}\, e^{ipx} \tilde{\psi}_n^{(\pm)}(p). \tag{27}$$

Since $\tilde{\psi}_n^{(+)}(p) (\tilde{\psi}_n^{(-)}(p))$ has a pole of order $n + 1$ at $p = i\omega (p = -i\omega)$, the eigenfunction $\psi_n^{(+)}(x) (\psi_n^{(-)}(x))$ has support only on positive(negative) x. Hence we find

$$\begin{aligned}\psi_n^{(+)}(x) =&\frac{(-1)^n}{n!} \theta(x) \sqrt{2\omega}\, \frac{d^n}{dp^n}\bigg|_{p=i\omega} \left[e^{ipx}(p + i\omega)^n\right] \\ =&(-1)^n \sqrt{2\omega}\theta(x) e^{-\omega x} L_n(2\omega x),\end{aligned} \tag{28}$$

where $L_n(z)$ designates a Laguerre polynomial of the n-th order and $\theta(x)$ is the step function. In the same manner, by calculating the residue at $p = -i\omega$, we obtain

$$\psi_n^{(-)}(x) = (-1)^n \sqrt{2\omega}\theta(-x)e^{\omega x} L_n(-2\omega x). \tag{29}$$

In short, we may write

$$\psi_n^{(\pm)}(x) = (-1)^n \sqrt{2\omega}\theta(\pm x)e^{-\omega r} L_n(2\omega r). \tag{30}$$

In this way, by writing the kernel $\langle x_b|e^{-\hat{H}_S\tau}|x_a\rangle$ as $K(x_b, \tau_b|x_a, \tau_a)$, we obtain

$$K(x_b, \tau_b|x_a, \tau_a) = \sum_{n=0}^{\infty} e^{-(n+1/2)\omega\tau} \left\{ \psi_n^{(+)}(x_b)\psi_n^{(+)}(x_a)^* + \psi_n^{(-)}(x_b)\psi_n^{(-)}(x_a)^* \right\}, \tag{31}$$

corresponding to Eq. (25).

We have thus observed that a time-sliced path integral,

$$\int \prod_{i=0}^{N} \frac{dp_i}{2\pi} \prod_{j=1}^{N} dx_j \, \exp\left[i \sum_{k=0}^{N} p_k(x_{k+1} - x_k) - \frac{\epsilon}{2} \sum_{k=1}^{N} \left(p_k p_{k-1} + \omega^2 \right) r_k \right] \tag{32}$$

that describes a kernel $K(x_b, \tau_b|x_a, \tau_a)$ in the limit $N \to \infty$, can be evaluated exactly by making multiple use of Cauchy's integral theorem. Therefore we can regard the model considered here an interesting example of an exact path integral that can be carried out without use of the Gaussian identity.

2.2 Path integral for the positive energy

So far our consideration has been restricted to the Hamiltonian $\hat{H}_S = \hat{p}\hat{r}\hat{p}/2 - E\hat{r}$ with a negative energy $E = -\omega^2/2$. It will be also interesting to see how the path integral of this system with a positive energy $E = k^2/2 (k > 0)$ can be solved. Hence our next target to be examined here is the time-sliced path integral

$$\int \prod_{i=0}^{N} \frac{dp_i}{2\pi} \prod_{j=1}^{N} dx_j \, \exp\left[i \sum_{j=0}^{N} p_j(x_{j+1} - x_j) - \frac{i\epsilon}{2} \sum_{j=1}^{N} \left(p_j p_{j-1} - k^2 \right) r_j \right]. \tag{33}$$

Unfortunately, however, we cannot define its Euclidean version without restricting the range of values for p as well as for the imaginary time $\tau = -it$ due to the lack of positivity of the Hamiltonian in the term proportional to $p^2 - k^2$. We shall therefore briefly point out some properties of the Euclidean path integral for this system at the end of this subsection.

We begin again with evaluation of the kernel in the momentum representation. By adding a positive infinitesimal to the imaginary part of k^2, we can perform integration over x in Eq.(33) to obtain a short-time kernel

$$\langle p| \left(1 - i\epsilon \hat{H}_S \right) |p'\rangle = \frac{1}{2\pi i} \left[\frac{1}{(p - p') + \epsilon \left(pp' - k^2 \right)/2} - \frac{1}{(p - p') - \epsilon \left(pp' - k^2 \right)/2} \right]. \tag{34}$$

Thanks to the regularization factor, poles of each term in this kernel are shifted off to the imaginary axis from the real line. For instance, as a function of p, the kernel $\langle p| \left(1 - i\epsilon \hat{H}_S \right) |p'\rangle$ has poles at

$$p = \frac{p' + \epsilon k^2/2}{1 + \epsilon p'/2} \quad \text{and} \quad \frac{p' - \epsilon k^2/2}{1 - \epsilon p'/2}, \tag{35}$$

whose imaginary part being positive and negative, respectively. Once we recognize this pole structure of the infinitesimal kernel, we can immediately carry out integrations over p for multiplication of short-time kernels by making use of the similar technique we have employed for the case of negative energy. We thus omit showing here the detail of the procedure for finding a Feynman kernel

$$\tilde{K}(p_b, t_b|p_a, t_a) = \sum_{\sigma=\pm 1} \frac{1}{2\pi i} \frac{\sigma}{(p_b - p_a)\cosh(kt/2) + \frac{\sigma}{k}\left(p_b p_a - k^2 \right) \sinh(kt/2)} \tag{36}$$

in the momentum representation for a finite time $t = t_b - t_a > 0$. Note that the regulator for a short-time kernel is included even in this expression as infinitesimal shifts $\mp i0_+$ for $\sigma = \pm 1$ in the denominator.

If we set $p = k\coth\theta$ or $p = k\tanh\theta$ corresponding to the sign of $p^2 - k^2$, it becomes easy to decompose the above kernel into an integration over the continuous eigenvalue of $\hat{H}_{\rm S}$. At the outset, we set $p_a = k\tanh\theta_a$ and $p_b = k\tanh\theta_b$. Then we rewrite each term in the kernel to obtain

$$\frac{1}{(p_b - p_a)\cosh(kt/2) \pm \dfrac{1}{k}(p_b p_a - k^2)\sinh(kt/2)} = \frac{\cosh\theta_b \cosh\theta_a}{k\sinh(\theta_b - \theta_a \mp kt/2)}. \tag{37}$$

A useful formula for our aim here is an integral representation

$$\frac{1}{\sinh x} = 2i\int_{-\infty}^{\infty} dz\, \frac{1}{1+e^{2\pi z}} e^{2ixz}, \tag{38}$$

in which a condition, $-\pi < \Im(x) < 0$, for the convergence being necessary. We here remember infinitesimal shifts $\mp i0_+$ in the Feynman kernel (36) to regard $\pm(\theta_b - \theta_a) - kt/2$ as x in Eq.(38). Then by noticing that $\sqrt{k^2 - p^2} = k/\cosh\theta$ for $p = k\tanh\theta$, we can express the kernel Eq.(36) as

$$\begin{aligned}\tilde{K}(p_b, t_b|p_a, t_a) &= \frac{k}{\pi}\frac{1}{\sqrt{(k^2 - p_b^2)(k^2 - p_a^2)}} \\ &\times \sum_{\sigma=\pm 1}\int_{-\infty}^{\infty} d\nu\, e^{-ikt\nu}\frac{1}{1+e^{2\pi\nu}}\left(\frac{k+p_b}{k-p_b}\right)^{i\sigma\nu}\left(\frac{k+p_a}{k-p_a}\right)^{-i\sigma\nu}\end{aligned} \tag{39}$$

for a combination of p_a and p_b that satisfy $|p_a| < k$ and $|p_b| < k$.

In the same way, for p_a and p_b satisfying $|p_a| > k$, $|p_b| > k$ as well as $p_a p_b - k^2 > 0$, we may set $p_a = k\coth\theta_a$ and $p_b = k\coth\theta_b$ to find

$$\frac{1}{(p_b - p_a)\cosh(kt/2) \pm \dfrac{1}{k}(p_b p_a - k^2)\sinh(kt/2)} = \frac{\sinh\theta_b \sinh\theta_a}{-k\sinh(\theta_b - \theta_a \mp kt/2)}, \tag{40}$$

in which $\theta_b - \theta_a$ should be regarded as $\theta_b - \theta_a \pm i0_+$ corresponding to the sign in front of $kt/2$. We then utilize the formula (38) again to obtain

$$\begin{aligned}\tilde{K}(p_b, t_b|p_a, t_a) &= \frac{k}{\pi}\frac{1}{\sqrt{(p_b^2 - k^2)(p_a^2 - k^2)}} \\ &\times \sum_{\sigma=\pm 1}\int_{-\infty}^{\infty} d\nu\, e^{-ikt\nu}\frac{e^{2\pi\nu}}{1+e^{2\pi\nu}}\left(\frac{p_b+k}{p_b-k}\right)^{i\sigma\nu}\left(\frac{p_a+k}{p_a-k}\right)^{-i\sigma\nu}.\end{aligned} \tag{41}$$

for this case. Here we have made a change of variable from ν to $-\nu$ and rewrite $1/(1+e^{-2\pi\nu})$ as $e^{2\pi\nu}/(1+e^{2\pi\nu})$ to reach this expression. Although there exist other possible combinations of values for p_a and p_b, it will be enough to observe Eq.(39) as well as Eq.(41) for finding eigenvalues and eigenfunctions, $\tilde{\psi}_\lambda^{(\pm)}(p) = \langle p|\lambda;\pm\rangle$, of $\hat{H}_{\rm S}$ in the momentum representation. Explicitly, wave functions are given by

$$\tilde{\psi}_\lambda^{(\pm)}(p) = \frac{1}{\sqrt{\pi(1+e^{2\pi\nu})}}\frac{(k+p)^{(-1\pm 2i\nu)/2}}{(k-p)^{(1\pm 2i\nu)/2}} \tag{42}$$

so that $|\lambda;\pm\rangle$ to be eigenvectors of $\hat{H}_{\rm S}$ with an eigenvalue $\lambda = k\nu$;

$$\hat{H}_{\rm S}|\lambda;\pm\rangle = k\nu|\lambda;\pm\rangle. \tag{43}$$

Note that the regularization for the short-time kernel Eq.(34) can now be interpreted as the prescription for avoiding branch points on the complex p-plane. In the region $|p| < k$, p is shifted with an amount of $\mp i0_+$ in $\tilde{\psi}_\lambda^{(\pm)}(p)$ by the effect of the regulator. On the other hand, when p is in regions where $|p| > k$, the shifts occur in the opposite way. This specifies the

way to go around the branch points at $p = \pm k$. When p moves along the real axis, a detour that goes lower and upper side of $p = \pm k$ is made for $\tilde{\psi}_\lambda^{(+)}(p)$ and for $\tilde{\psi}_\lambda^{(-)}(p)$, respectively. Due to these singularities of branch points, discontinuities of wave functions as functions of a real variable p at $p = \pm k$ are unavoidable; if we consider the analytical continuation of the eigenfunction $\psi_\lambda^{(+)}(p)$ in the region $|p| < k$ to that of $p > k$, for instance, the wave function gets a factor $e^{-i\pi/2+\pi\nu}$ multiplied in addition to a switch from $k - p$ to $p - k$ in the denominator. Nevertheless, we can clarify the completeness of the eigenfunctions simply by setting $t = 0$ in each expression, Eq.(39), (41), and in other similar ones, of the Feynman kernel. If we keep in mind the discontinuity of eigenfunctions at branch points, we can also easily check the following normalization and the orthogonality relations of $|\lambda; \pm\rangle$,

$$\langle \lambda; \sigma | \lambda'; \sigma' \rangle = \delta_{\sigma,\sigma'} \delta(\lambda - \lambda'). \tag{44}$$

We have thus completed the calculation of the Feynman kernel in the momentum representation.

Turning now to the original time-sliced path integral of Eq.(33) that defines a Feynman kernel in the position diagonal representation, we make use of the integral representation of the confluent hypergeometric, or Kummer, function to obtain

$$\frac{(k+p)^{(-1+2i\nu)/2}}{(k-p)^{(1+2i\nu)/2}} = e^{\pi\nu} \int_0^\infty dx\, e^{-i(p-k)x} F(\frac{1}{2} - i\nu, 1, -2ikx), \tag{45}$$

where the shift of p by an amount of $-i0_+$ must be understood. With the help of this formula, we can find the eigenfunction, $\langle x|\lambda; +\rangle = \psi_\lambda^{(+)}(x)$, for an eigenvalue $\lambda = k\nu$ in the position representation as

$$\psi_\lambda^{(+)}(x) = \theta(x) \sqrt{\frac{e^{\pi\nu}}{\cosh \pi\nu}} e^{ikx} F(\frac{1}{2} - i\nu, 1, -2ikx). \tag{46}$$

In the same way, $\langle p|\lambda; -\rangle$ can be transformed to $\langle x|\lambda; -\rangle = \psi_\lambda^{(-)}(x)$ as

$$\psi_\lambda^{(-)}(x) = \theta(-x) \sqrt{\frac{e^{\pi\nu}}{\cosh \pi\nu}} e^{-ikx} F(\frac{1}{2} - i\nu, 1, 2ikx). \tag{47}$$

These results enable us to transform the Feynman kernel $\tilde{K}(p_b, t_b|p_a, t_a)$ in the momentum representation to the one in the position representation. Writing the Fourier transform of $\tilde{K}(p_b, t_b|p_a, t_a)$ as $K(x_b, t_b|x_a, t_a)$, we finally observe that the time-sliced path integral of Eq.(33) can be carried out exactly to yield

$$K(x_b, t_b|x_a, t_a) = \int_{-\infty}^\infty d\lambda\, e^{-i\lambda t} \left\{ \psi_\lambda^{(+)}(x_b) \psi_\lambda^{(+)}(x_a)^* + \psi_\lambda^{(-)}(x_b) \psi_\lambda^{(-)}(x_a)^* \right\}. \tag{48}$$

It will be obvious that the essence of the exact calculation in the time-sliced path integral is again the pole structure of the short-time kernel Eq.(34) similar to the one we have found in Eq.(12) for the case of a negative energy $E = -\omega^2/2$.

As we have stated above at the beginning of this subsection, the Euclidean path integral of the Hamiltonian $\hat{H}_{\rm S}$ with a positive energy $E = k^2/2$ cannot be freely defined. Nevertheless, we may try to set $t = -i\tau$(Wick-rotation) in expressions, Eq.(39) and (41), of the Feynman kernel $\tilde{K}(p_b, t_b|p_a, t_a)$ in the momentum representation. If we make this change in Eq.(41), we find a condition, $0 < k\tau < 2\pi$, for the convergence of the integration over ν. On the other hand, Eq.(39) can be Wick-rotated only for the case $-2\pi < k\tau < 0$ holds. Hence a kernel, though existed, for the imaginary time τ cannot be extended its domain of definition into other regions of momentum variables. For this reason we are restricted to make considerations on Feynman kernels with t in the real domain.

3 Physical States

We have clarified in the previous section that path integrals of the Hamiltonian $\hat{H}_{\rm S}$ for Feynman kernels with both positive and negative energies can be performed in an exact manner. The Feynman kernel of $\hat{H}_{\rm S}$ is, however, the one for the unphysical system in a specific gauge. To get

information of the original system, we need to take the physical state condition (4) into account. This can be achieved by considering the resolvent operator $(\hat{H}-E)^{-1}$, where $\hat{H}$ expresses the Hamiltonian of the original system without gauge invariance. In this section, we first examine the resolvent operator for a free particle with a negative energy $E=-\omega^2/2$, then generalize the method to fit the Coulomb system by making use of the result in the previous section. Establishing the prescription to find the resolvent operator from the Feynman kernel in these examples, we eventually solve the scattering states of the one-dimensional Coulomb system.

For the case of a free particle governed by the Hamiltonian $\hat{H}=\hat{p}^2/2$, we first define non-symmetric operator $\hat{H}_{\rm R}=(\hat{H}-E)\hat{r}(E=-\omega^2/2)$ to observe

$$\left(\hat{H}-E\right)^{-1}=\hat{r}\hat{H}_{\rm R}^{-1}. \tag{49}$$

To evaluate the right hand side of the above in terms of the path integral, we here invent a useful relation that connects $\hat{H}_{\rm R}$ to the symmetric one $\hat{H}_{\rm S}$. To this aim we introduce an operator $\hat{\sigma}$ by $\hat{\sigma}=\hat{x}/\hat{r}$, which can be defined excepting the origin $x=0$, to express the sign of the position operator. Note that it is also possible to introduce $\hat{\sigma}$ by the commutator $[\hat{r},\hat{p}]=i\hat{\sigma}$ as we have already done implicitly in obtaining Eq.(11). As an important fact, it should be remarked here that $\hat{\sigma}$ commutes with $\hat{H}_{\rm S}=\hat{p}\hat{r}\hat{p}/2-E\hat{r}$, i.e. $\hat{\sigma}\hat{H}_{\rm S}=\hat{H}_{\rm S}\hat{\sigma}$. It will be clear that the operator $\hat{\sigma}$ is responsible for the supplementary quantum number σ in the eigenfunctions of the Hamiltonian $\hat{H}_{\rm S}$. Needless to say, it is not a good quantum number for Hamiltonians other than $\hat{H}_{\rm S}$. Nevertheless it will be useful for finding eigenfunctions of the Coulomb Hamiltonian by restricting the domain of x to one of a half line corresponding to the value of σ. Keeping these in mind, we now define another operator $\omega-i\hat{p}$ and its Hermitian conjugate to find

$$\hat{H}_{\rm R}=(\omega\mp i\hat{p})\left(\hat{H}_{\rm S}\pm\frac{\omega}{2}\hat{\sigma}\right)(\omega\mp i\hat{p})^{-1}. \tag{50}$$

Though both of these formulas are equally useful for our purpose, we here choose the one of $\omega-i\hat{p}$ for the formulation below.

By writing $\tilde{\psi}_n^{(\pm)}(p)(n=0,\,1,\,2,\,\dots)$ in Eq.(26) as $\langle p|n;\pm\rangle$, we denote right eigenvectors of $\hat{H}_{\rm S}$ as $|n;\pm\rangle$. Then we define $|\psi_{\rm R}^{(n;\pm)}\rangle=(\omega-i\hat{p})|n;\pm\rangle$ to find

$$e^{-\hat{H}_{\rm R}\tau}|\psi_{\rm R}^{(n;+)}\rangle=e^{-(n+1)\omega\tau}|\psi_{\rm R}^{(n;+)}\rangle,\quad e^{-\hat{H}_{\rm R}\tau}|\psi_{\rm R}^{(n;-)}\rangle=e^{-n\omega\tau}|\psi_{\rm R}^{(n;-)}\rangle. \tag{51}$$

Completeness of the eigenvectors of $\hat{H}_{\rm S}$ will be expressed in terms of $|\psi_{\rm R}^{(n;\pm)}\rangle$ and their conjugates, defined by $\langle\bar{\psi}_{\rm R}^{(n;\pm)}|=\langle n;\pm|(\omega-i\hat{p})^{-1}$, to yield

$$e^{-\hat{H}_{\rm R}\tau}=\sum_{n=0}^{\infty}\left\{e^{-(n+1)\omega\tau}|\psi_{\rm R}^{(n;+)}\rangle\langle\bar{\psi}_{\rm R}^{(n;+)}|+e^{-n\omega\tau}|\psi_{\rm R}^{(n;-)}\rangle\langle\bar{\psi}_{\rm R}^{(n;-)}|\right\}. \tag{52}$$

Sandwiching between $\langle p_b|$ and $|p_a\rangle$ this expression after multiplying $\hat{r}$ from the left, we obtain

$$\begin{aligned}&\langle p_b|\hat{r}e^{-\hat{H}_{\rm R}\tau}|p_a\rangle\\ =&\sum_{n=0}^{\infty}\left\{e^{-(n+1)\omega\tau}\langle p_b|\hat{r}|\psi_{\rm R}^{(n;+)}\rangle\langle\bar{\psi}_{\rm R}^{(n;+)}|p_a\rangle+e^{-n\omega\tau}\langle p_b|\hat{r}|\psi_{\rm R}^{(n;-)}\rangle\langle\bar{\psi}_{\rm R}^{(n;-)}|p_a\rangle\right\}.\end{aligned} \tag{53}$$

Evaluation of the operator $\hat{r}$ in the above is done as follows: we may write $\hat{r}=\hat{x}\hat{\sigma}$ in which $\hat{x}$ has clear definition in the momentum representation, given by $\langle p|\hat{x}=i\partial\langle p|/\partial p$, our task reduces to calculate $\langle p|\hat{\sigma}|\psi_{\rm R}^{(n;\sigma)}\rangle$ for $\sigma=\pm1$. By noticing $[\hat{\sigma},\hat{p}]=2i\delta(\hat{x})$, where the delta function of $\hat{x}$ being defined by

$$\delta(\hat{x})=\frac{1}{2\pi}\int_{-\infty}^{\infty}dp'\,e^{ip'\hat{x}} \tag{54}$$

so that

$$\langle p|\delta(\hat{x})=\frac{1}{2\pi}\int_{-\infty}^{\infty}dp'\,\langle p'|,\quad\langle p'|\hat{p}=p'\langle p'|, \tag{55}$$

we rewrite $\langle p|\hat{\sigma}|\psi_{\rm R}^{(n;\sigma)}\rangle$ as

$$\langle p|\left\{(\omega-i\hat{p})\hat{\sigma}-i[\hat{\sigma},\hat{p}]\right\}|n;\sigma\rangle=\sigma(\omega-ip)\langle p|n;\sigma\rangle+\frac{1}{\pi}\int_{-\infty}^{\infty}dp'\,\langle p'|n;\sigma\rangle. \tag{56}$$

Since the second term above is independent of p to be destroyed by differentiation with respect to p, only the first term contributes to $\langle p|\hat{r}|\psi_{\rm R}^{(n;\sigma)}\rangle$ to yield

$$\langle p|\hat{r}|\psi_{\rm R}^{(n;\pm)}\rangle = \pm i\frac{\partial}{\partial p}\left\{(\omega - ip)\langle p|n;\pm\rangle\right\}. \tag{57}$$

Remembering the definition of wave functions $\tilde{\psi}_n^{(\pm)}(p) = \langle p|n;\pm\rangle$ in Eq.(26), we may write explicitly as

$$\begin{aligned}\langle p|\hat{r}|\psi_{\rm R}^{(n;+)}\rangle =& \sqrt{\frac{\omega}{\pi}}\frac{2(n+1)\omega}{\omega^2+p^2}\left(\frac{\omega-ip}{\omega+ip}\right)^{n+1},\\ \langle p|\hat{r}|\psi_{\rm R}^{(n;-)}\rangle =& \sqrt{\frac{\omega}{\pi}}\frac{2n\omega}{\omega^2+p^2}\left(\frac{\omega+ip}{\omega-ip}\right)^{n}.\end{aligned} \tag{58}$$

Here it must be remarked that $\langle p|\hat{r}|\psi_{\rm R}^{(n;-)}\rangle$ vanishes for $n=0$ thereby disappearing from the series in Eq.(53). Combining the result above with those of $\langle\bar{\psi}_{\rm R}^{(n;\pm)}|p_a\rangle$, we find

$$\begin{aligned}\langle p_b|\hat{r}e^{-\hat{H}_{\rm R}\tau}|p_a\rangle =& \sum_{n=1}^{\infty} e^{-n\omega\tau}\frac{\omega}{\pi}\frac{2n\omega}{(\omega^2+p_b^2)(\omega^2+p_a^2)}\\ &\times\left\{\left(\frac{\omega-ip_b}{\omega+ip_b}\frac{\omega+ip_a}{\omega-ip_a}\right)^n+\left(\frac{\omega+ip_b}{\omega-ip_b}\frac{\omega-ip_a}{\omega+ip_a}\right)^n\right\}.\end{aligned} \tag{59}$$

We are now at the point to observe, by integrating the above with respect to τ, that

$$\langle p_b|\frac{1}{\hat{H}-E}|p_a\rangle = \frac{2}{(\omega^2+p_b^2)}\delta(p_b-p_a) - \frac{\omega}{\pi}\frac{2}{(\omega^2+p_b^2)(\omega^2+p_a^2)} \tag{60}$$

for $E=-\omega^2/2$. It is precisely the first term of the right hand side that should be defined by the left hand side. Therefore the second term above indicates a contradiction in our manipulation. The origin of this obvious disagreement is the disappearance of $\langle p|\hat{r}|\psi_{\rm R}^{(0;-)}\rangle$ from the sum. A quick recovery for this discrepancy will be possible if we keep $\langle p|\hat{r}|\psi_{\rm R}^{(0;-)}\rangle$ until we carry out the integration over τ, by replacing it with $e^{-\nu\omega\tau}\langle p|\hat{r}|\psi_{\rm R}^{(\nu;-)}\rangle$, thereupon letting $\nu\to 0$. To be consistent, however, we should examine this phenomenon in detail otherwise we will not be successful in establishing the method to define the operator $\hat{r}\hat{H}_{\rm R}^{-1}$. Though interesting, it will not be so useful to look more closely at the resolvent operator with a negative energy for the free particle. Hence we terminate the analysis on this system here by learning that the definition of the resolvent operator by its action on each eigenfunction of the non-symmetric operator $\hat{H}_{\rm R}$ fails due to the disappearance of an eigenfunction from the complete set.

We now switch the Hamiltonian to that of the Coulomb system by changing $\hat{H}_{\rm R}$ to $\hat{H}_{\rm R}-\alpha$ in the above argument. Observation of the failure in the naive approach above by integration over τ term by term on each eigenfunction of $\hat{H}_{\rm R}$ suggests us to try dealing with each term in the Euclidean kernel

$$\begin{aligned}&\langle p_b|e^{-(\hat{H}_{\rm S}-\alpha)\tau}|p_a\rangle\\ =&\sum_{\sigma=\pm1}\frac{1}{2\pi i}\frac{\sigma}{(p_b-p_a)\cosh(\omega\tau/2) - i\dfrac{\sigma}{\omega}\left(p_bp_a+\omega^2\right)\sinh(\omega\tau/2)}e^{\alpha\tau}\end{aligned} \tag{61}$$

as a whole. To this aim, let us set $p_b=\omega\tan\theta_b$ and $p_a=\omega\tan\theta_a$ then rewrite the above in terms of these new variables. Firstly, we write

$$\begin{aligned}&(p_b-p_a)\cosh(\omega\tau/2)\mp\frac{i}{\omega}\left(p_bp_a+\omega^2\right)\sinh(\omega\tau/2)\\ =&\frac{\omega}{\cos\theta_b\cos\theta_a}\sin(\theta_b-\theta_a\mp i\omega\tau/2)\end{aligned} \tag{62}$$

to find

$$\begin{aligned}&\langle p_b|\hat{r}e^{-(\hat{H}_{\rm R}-\alpha)\tau}|p_a\rangle\\ =&\sum_{\sigma=\pm1}\frac{\cos^2\theta_b\cos^2\theta_a}{2\pi\omega^2}\frac{\partial}{\partial\theta_b}\frac{e^{-i(\theta_b-\theta_a-i\sigma\omega\tau/2)+\alpha\tau}}{\sin(\theta_b-\theta_a-i\sigma\omega\tau/2)}\\ =&-\frac{\cos^2\theta_b\cos^2\theta_a}{2\pi\omega^2}\left[\frac{e^{\alpha\tau}}{\sin^2(\theta_b-\theta_a-i\omega\tau/2)}+\frac{e^{\alpha\tau}}{\sin^2(\theta_b-\theta_a+i\omega\tau/2)}\right].\end{aligned} \tag{63}$$

Since we are dealing with the case $\alpha > 0$(attractive Coulomb potential), both of the terms in the above monotonically tends to 0 if we let $\tau \to -\infty$. Although the path integral for the Euclidean kernel has been performed only for $\tau > 0$, the result can be regarded as a function of a complex variable τ, besides being a function of p_b and p_a. The above mentioned behavior in the asymptotic region, $\tau \to -\infty$, naturally suggests us to integrate Eq.(63) with respect to τ form 0 to $-\infty$ for a definition of $\hat{r}(\hat{H}_{\rm R} - \alpha)^{-1}$. However, when the interaction is turned off by setting $\alpha = 0$, the behavior of the first term becomes wrong. This will be confirmed easily by looking at the second line in the above for $\sigma = +1$ and $\tau \to -\infty$. Such a discrepancy must be avoided for the consistency of the formalism. We therefore choose the domain for integration over τ from 0 to $+\infty$ for $\sigma = +1$ and from 0 to $-\infty$ for $\sigma = -1$ with the assumption $\omega > \alpha$ for a while. Note that, by this new prescription, we can explain the inconsistency found in Eq.(60) above. The undesirable term is the consequence of the non-vanishing, though finite, behavior of the second term in Eq.(63) when integrated over τ from 0 to $+\infty$($\alpha = 0$ should be set for the free particle).

We here utilize the integral representation for a hypergeometric function to carry out the integration of Eq.(63) over τ, thereby obtaining

$$\int_0^\infty \frac{e^{\pm\alpha\tau}\, d\tau}{\sin^2(\theta_b - \theta_a - i\omega\tau/2)} = -\frac{4}{\omega}\frac{e^{-2i(\theta_b-\theta_a)}}{1 \mp \hat{\alpha}} F(2, 1 \mp \hat{\alpha}, 2 \mp \hat{\alpha}; e^{-2i(\theta_b-\theta_a)}), \tag{64}$$

where $\hat{\alpha} = \alpha/\omega$ and $F(a, b, c; z)$ denotes a hypergeometric function. Here we should recall the initial condition, $S_0 = 0_+$, imposed in the derivation of the Euclidean kernel. We have chosen 0_+ instead of 0 to make the kernel reproduce the delta function $\delta(p_b - p_a)$ even after taking the limit $\tau \to 0$. This prescription now takes effect in the above by multiplying $e^{\mp 0_+}$ to $e^{-2i(\theta_b-\theta_a)}$ depending on the sign in front of $\hat{\alpha}$. In view of this prescription, we should rewrite one of the above as

$$-\frac{z}{1+\hat{\alpha}} F(2, 1+\hat{\alpha}, 2+\hat{\alpha}; z) = \frac{1}{1-\hat{\alpha}}\frac{1}{z} F(2, 1-\hat{\alpha}, 2-\hat{\alpha}; \frac{1}{z}) + \frac{\pi\hat{\alpha}}{\sin\pi\hat{\alpha}}(-z)^{-\hat{\alpha}} \tag{65}$$

for $z = e^{-2i(\theta_b-\theta_a+i0_+)}$ to be useful as a hypergeometric series. Taking these into account, we obtain

$$\begin{aligned}
&\langle p_b|\frac{1}{\hat{H}_{\rm C} - E - i0_+}|p_a\rangle = \frac{\omega}{\pi}\frac{2}{(\omega^2 + p_b^2)(\omega^2 + p_a^2)} \\
&\times \left\{ \frac{1}{1-\hat{\alpha}} \left(e^{-2i(\theta_b-\theta_a-i0_+)} F(2, 1-\hat{\alpha}, 2-\hat{\alpha}; e^{-2i(\theta_b-\theta_a-i0_+)}) \right.\right. \\
&\left. + e^{2i(\theta_b-\theta_a+i0_+)} F(2, 1-\hat{\alpha}, 2-\hat{\alpha}; e^{2i(\theta_b-\theta_a+i0_+)}) \right) \\
&\left. + \frac{\pi\hat{\alpha}}{\sin\pi\hat{\alpha}} \left(-e^{-2i(\theta_b-\theta_a+i0_+)} \right)^{-\hat{\alpha}} \right\}
\end{aligned} \tag{66}$$

for the Hamiltonian $\hat{H}_{\rm C} = \hat{p}^2/2 - \alpha/\hat{r}$ with $E = -\omega^2/2$. Here use has been made of the fact that the effect of our prescription of adding $\pm 0_+$ in the denominator of the kernel is equivalent to regarding $E = -\omega^2/2$ as to be associated with $+i0_+$. Series expansions of hypergeometric functions in the above will then bring us

$$\begin{aligned}
&\langle p_b|\frac{1}{\hat{H}_{\rm C} - E - i0_+}|p_a\rangle = \frac{\omega}{\pi}\frac{2}{(\omega^2 + p_b^2)(\omega^2 + p_a^2)} \\
&\times \left[\sum_{n=1}^{\infty} \frac{n}{n-\hat{\alpha}} \left\{ \left(\frac{\omega - ip_b}{\omega + ip_b}\frac{\omega + ip_a}{\omega - ip_a} \right)^n + \left(\frac{\omega + ip_b}{\omega - ip_b}\frac{\omega - ip_a}{\omega + ip_a} \right)^n \right\} \right. \\
&\left. + \frac{2\pi i\hat{\alpha}}{1 - e^{-2\pi i\hat{\alpha}}} \left(\frac{\omega - ip_b}{\omega + ip_b}\frac{\omega + ip_a}{\omega - ip_a} \right)^{-\hat{\alpha}} \right].
\end{aligned} \tag{67}$$

Clearly, there exist poles of the Green function (67) at $E = E_n$,

$$E_n = -\frac{\omega_n^2}{2},\ \omega_n = \frac{\alpha}{n}, \tag{68}$$

for $n = 1, 2, 3, \ldots$, corresponding to bound states with twofold degeneracy for each energy eigenvalue. By approximating $\omega(E) = \sqrt{-2E}$ as $\omega(E) \simeq \omega_n + (E_n - E)/\omega_n$ in the vicinity of n-th eigenvalue, we obtain

$$\begin{aligned}&\frac{\omega}{\pi}\frac{2}{(\omega^2+p_b^2)(\omega^2+p_a^2)}\frac{n}{n-\hat{\alpha}}\left(\frac{\omega\mp ip_b}{\omega\pm ip_b}\frac{\omega\pm ip_a}{\omega\mp ip_a}\right)^n\\ \simeq&\frac{1}{E_n-E}\frac{2}{\pi}\left(\frac{\alpha}{n}\right)^3\frac{1}{(\omega_n^2+p_b^2)(\omega_n^2+p_a^2)}\left(\frac{\omega_n\mp ip_b}{\omega_n\pm ip_b}\frac{\omega_n\pm ip_a}{\omega_n\mp ip_a}\right)^n\end{aligned} \tag{69}$$

to extract the pole parts of the Green function as

$$\langle p_b|\frac{1}{\hat{H}_{\rm C}-E-i0_+}|p_a\rangle \simeq \sum_{n=1}^{\infty}\frac{1}{E_n-E}\left\{\tilde{\psi}^{(+)}_{{\rm C},n}(p_b)\tilde{\psi}^{(+)}_{{\rm C},n}(p_a)^* + \tilde{\psi}^{(-)}_{{\rm C},n}(p_b)\tilde{\psi}^{(-)}_{{\rm C},n}(p_a)^*\right\} \tag{70}$$

where

$$\tilde{\psi}^{(\pm)}_{{\rm C},n}(p) = \sqrt{\frac{2}{\pi}}\left(\frac{\alpha}{n}\right)^{3/2}\frac{1}{\omega_n^2+p^2}\left(\frac{\omega_n\mp ip}{\omega_n\pm ip}\right)^n. \tag{71}$$

We have thus clarified the pole structure of the Green function. Apart from the continuous spectrum, poles of the Green function precisely describe the zeros of $\hat{H}_{\rm C} - E$. The gauge fixing condition, given by $H_{\rm T} \simeq 0$, in our formulation of the corresponding classical system is proportional to $H_{\rm C} - E$ with an almost non-vanishing factor r. Therefore we may regard enforcing $\hat{H}_{\rm C} - E = 0$ in quantum mechanics as the implementation of the gauge fixing condition that picks up physical or admissible states from the Hilbert space of the system with a gauge degree.

Another Green function that reflects different boundary condition is obtained by changing the prescription from $E \to E + i0_+$ to $E \to E - i0_+$, instead. For this case, poles appear at $\omega = -\omega_n$. These cannot be regarded as bound states since $\Re(\omega) \geq 0$ is needed in setting $E = -\omega^2/2$. Wave functions for continuous eigenvalues can be found by subtracting $(\hat{H}_{\rm C} - E + i0_+)^{-1}$ from $(\hat{H}_{\rm C} - E - i0_+)^{-1}$. It will be useful here to consider in the region $E = k^2/2 \geq 0$ to find the continuous eigenfunctions. The transition from a negative energy to a positive one can be most easily achieved by setting $\omega \to -ik$, $\theta_{b,a} \to i\theta_{b,a}$ as well as writing α/k as $\hat{\alpha}$ again. The expression for $E = -\omega^2/2$ in Eq.(63) then becomes

$$\begin{aligned}&\langle p_b|\hat{r}e^{-i(\hat{H}_{\rm R}-\alpha)t}|p_a\rangle\\ =&-\frac{\cosh^2\theta_b\cosh^2\theta_a}{2\pi k^2}\left[\frac{e^{i\alpha t}}{\sinh^2(\theta_b-\theta_a-ikt/2)}+\frac{e^{i\alpha t}}{\sinh^2(\theta_b-\theta_a+ikt/2)}\right].\end{aligned} \tag{72}$$

This expression will be integrated over t from 0 to $+\infty(-\infty)$ for the first(second) term to yield

$$\begin{aligned}&\langle p_b|\frac{1}{\hat{H}_{\rm C}-E-i0_+}|p_a\rangle = \frac{k}{\pi}\frac{2}{(k^2-p_b^2)(k^2-p_a^2)}\\ &\times\int_{-\infty}^{\infty}\frac{d\nu}{e^{2\pi\nu}-1}\frac{\nu}{\nu-\hat{\alpha}}\left\{\left(\frac{k+p_b}{k-p_b}\frac{k-p_a}{k+p_a}\right)^{i\nu}+\left(\frac{k-p_b}{k+p_b}\frac{k+p_a}{k-p_a}\right)^{i\nu}\right\},\end{aligned} \tag{73}$$

in which $\hat{\alpha} = \alpha/k$ has a negative imaginary part due to the boundary condition of the Green function: $k \to k + i0_+$ for $E \to E + i0_+$. Here we have made use of the formula Eq.(38) again in obtaining the above. If we change the boundary condition by replacing this with $E \to E - i0_+$, the sign of the imaginary part of $\hat{\alpha}$ is also changed as well. Then, by subtraction of these two Green functions, we get a delta function of $\nu - \hat{\alpha}$ to find

$$\begin{aligned}&\frac{1}{2\pi i}\langle p_b|\left[\frac{1}{\hat{H}_{\rm C}-E-i0_+}-\frac{1}{\hat{H}_{\rm C}-E+i0_+}\right]|p_a\rangle = \frac{k}{\pi}\frac{2}{(k^2-p_b^2)(k^2-p_a^2)}\\ &\times\frac{\hat{\alpha}}{e^{2\pi\hat{\alpha}}-1}\left\{\left(\frac{k+p_b}{k-p_b}\frac{k-p_a}{k+p_a}\right)^{i\hat{\alpha}}+\left(\frac{k-p_b}{k+p_b}\frac{k+p_a}{k-p_a}\right)^{i\hat{\alpha}}\right\}\end{aligned} \tag{74}$$

by omitting the pole parts. From this expression, eigenfunctions for $E = k^2/2$ immediately read

$$\tilde{\psi}^{(\pm)}_{{\rm C},k}(p) = \sqrt{\frac{\alpha e^{-\pi\hat{\alpha}}}{\pi\sinh\pi\hat{\alpha}}}\frac{1}{k^2-p^2}\left(\frac{k\pm p}{k\mp p}\right)^{i\hat{\alpha}}. \tag{75}$$

Corresponding eigenfunctions in coordinate representation are also obtained by Fourier transformation as

$$\psi^{(\pm)}_{\mathrm{C},k}(x) = 2\sqrt{\frac{\pi\alpha e^{\pi\hat{\alpha}}}{\sinh\pi\hat{\alpha}}}\theta(\pm x)e^{ikr}rF(1-i\hat{\alpha},2,-2ikr),\ r=|x|. \tag{76}$$

In the same way, eigenfunctions for bound states, given by Eq.(71), can be transformed to those of coordinate representation to yield

$$\psi^{(\pm)}_{\mathrm{C},n}(x) = 2(-1)^{n+1}\omega_n^{3/2}\theta(\pm x)e^{-\omega_n r}rF(1-n,2,2\omega_n r),\ \omega_n=\frac{\alpha}{n},\ r=|x|. \tag{77}$$

Since σ is not suitable for description of the Coulomb system, we may consider linear combinations of wave functions above for construction of eigenfunctions with even and odd parities. This completes the process of solving the Coulomb system in terms of path integral.

To close this section let us add a comment on the difference in the number of bound states of the Coulomb system from that of H_{S}. We have observed that, through the procedure of constructing $(\hat{H}_{\mathrm{C}}-E)^{-1}$ from the eigenfunctions of $\hat{H}_{\mathrm{S}}$ with the aid of $\omega - i\hat{p}$ and its inverse, one of the eigenvectors of $\hat{H}_{\mathrm{S}}$ has been eliminated from the series in Eq.(53). Therefore the dimension of the space spanned by the bound states of the Coulomb system is fewer by this missing one than that of $\hat{H}_{\mathrm{S}}$ with $E=-\omega^2/2$. Since $\hat{H}_{\mathrm{S}}$ is Hermitian, its eigenvectors form a complete set. Hence the bound states of $\hat{H}_{\mathrm{C}}$ cannot be complete. This will become clear if we set $\alpha=0$, by which we return to the free particle, in Eq.(67). The sum over discrete eigenvalues is missing a term of $n=0$, thereby failing to reproduce a delta function $\delta(p_b-p_a)$. Interestingly, the missing term is supplied from the term of continuous eigenvalues. The same can be seen more clearly in the integral of Eq.(73) for the resolvent operator of a positive energy. For this case, if $\alpha\neq 0$, poles from $\nu/(e^{2\pi\nu}-1)$ for bound states are complemented by the contribution from the pole of $1/(\nu-\hat{\alpha})$ for continuous eigenvalues to form a complete set. If we set $\alpha=0$ after shifting ν by $-i/2(+i/2)$ for the first(second) term, the pole at $\nu=\hat{\alpha}$ is absorbed into those of $1/(1+e^{2\pi\nu})$ by adding new poles at $\nu=\pm i/2$. Cauchy's integral theorem then tells us that the sum over these poles at $\nu=\pm i(n+1/2)(n=0,\,1,\,2,\,\dots)$, with suitable regulators for each sum from upper and lower half-planes, is entirely equivalent to the integration of the continuous eigenfunctions over ν along the real axis, therefore resulting in a delta function $\delta(p_b-p_a)$ from both approach. It is exactly the pole at $\nu=\hat{\alpha}$ in Eq.(73) that generates the term of continuous eigenvalues in Eq.(67). Therefore, we are convinced the incompleteness of the set of eigenfunctions of bound states of the Coulomb system. It will be further interesting to notice that the eigenvalue, $E_0=-\infty$, of the missing discrete eigenfunction is connected to that of $E=k^2/2$ by letting $k\to\infty$ because, if $\alpha\neq 0$, $\hat{\alpha}=0$ is possible only in this limit. Namely, the scattering state with an energy of positive infinity is complementary to the bound state with an energy with negative infinity though unphysical they are. A very intuitive understanding for this will be possible if we remember the behaviors of eigenfunctions in the coordinate representation; both limits, $n\to 0$ so that $\omega_n\to-\infty$ for the bound state and $k\to\infty$ for the scattering state, yield a delta function $\delta(x)$ although approaching in different directions. Finally, the absence of the bound state for $n=0$ in Eq.(67) may be regarded as the consequence of the boundary condition imposed naturally by the formulation of the path integral for wave functions at $x=0$. In view of the behavior of eigenfunctions at $x=0$, we recognize that the path integral requires the Dirichlet condition at the origin.

4 Conclusions

On the basis of the Duru-Kleinert transformation with Fujikawa's gauge theoretical technique, we have formulated exact path integrals for the Coulomb potential in one space dimension. The mechanism of the exactness is explained by the pole structure of kernels in the momentum representation so that the time-sliced path integrals are evaluated simply by the use of Cauchy's integral theorem. Therefore the model considered here will be regarded as a new example of the path integral that can be performed exactly without making use of the Gaussian integration. It will be, however, difficult to observe such a structure if we employ the formal continuum formulation path integrals. A special care on the bound state with infinite binding energy has

been taken to clarify its absence. The disappearance of this unphysical state in the eigenfunction expansion of the resolvent operator should be attributed to a proper prescription for the boundary condition at the origin required by the path integral formalism.

The mechanism of the exact path integrals may have another interpretation; each relation between p_a and p_b read from the poles of the kernels in the momentum representation defines an action of the one parameter subgroup of $SL(2, \boldsymbol{R})$(setting $\tau = it$ is assumed for the case of a negative energy) through a linear fractional transformation. This will bring us a group theoretical approach for the understanding of the exactness. From this view point, generalizations of our technique to be useful for other systems will be interesting for future investigation.

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It is shown in Ref.[25] that, in principle, an infinite magnetic field confines the two particles of the electron-hole system(an exciton) to the axis parallel to the field direction and the system tends to a kind of 1-D H-atom.

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